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1981 J. Phys. A: Math. Gen. 14 2317

(http://iopscience.iop.org/0305-4470/14/9/025)

J. Phys. A: Math. Gen. 14 (1981) 2317–2323. Printed in Great Britain

New two- and three-parameter solutions of the MPST equation

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Received 8 October 1980, in final form 26 February 1981

Abstract. We present some new two- and three-parameter solutions of the MPST equation. All the three-parameter solutions are physical in the sense of asymptotic flatness. The simplest member of the three-parameter series of solutions is identical with a three-parameter solution of the static Einstein–Maxwell equations recently discovered by Bonnor.

1. Introduction

Misra *et al* (1973) have developed a new method for generating the Weyl class as well as a more general class of electromagnetic fields. They have shown that by the same procedure a new class of Maxwell fields can be generated which are not of the Weyl type. They have presented a particular solution of this class which represents an asymptotically flat gravitational field of a body possessing an electric or magnetic dipole moment. At large distances and in the case of the vanishing dipole parameter, the gravitational field goes over to the Schwarzschild field.

Starting from the solution mentioned above (we call it the MPST solution after Misra *et al*) and making use of Kinnersley's method of generating stationary Einstein–Maxwell fields from known solutions of Einstein–Maxwell equations, Esposito and Witten (1973) have obtained a five-parameter solution which describes a source containing mass, electric charge, magnetic dipole, higher multiple moments of all three kinds and angular momentum. This solution is also asymptotically flat.

Later Wang (1974) has presented some solutions of the MPST equation (briefly derived in § 2). In this paper we present some new two- and three-parameter solutions of the same equation. All the three-parameter solutions are physical in the sense of asymptotic flatness. It is interesting to note that the simplest of these solutions (charged MPST solution derived in § 4.1) is identical with a three-parameter solution of the static Einstein–Maxwell equations recently discovered by Bonnor (1979).

Before proceeding to work on the MPST equation, it is important to point out that Misra *et al* used the electromagnetic energy–momentum tensor in the form

$$E_{\mu\nu}=F_{\mu\alpha}F^{\alpha}_{\ \nu}-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{1}$$

instead of the usual form

$$E_{\mu\nu}=-F_{\mu\alpha}F^{\alpha}_{\ \nu}+\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{2}$$

0305-4470/81/092317+07\$01.50

Therefore, as pointed out by Ward (1974), the physical situation will be obtained by adopting the following transformations for the electromagnetic potential C, the dipole moment parameter e and the charge parameter γ

$$C \to \mathrm{i}C \qquad e \to \mathrm{i}e \qquad \gamma \to \mathrm{i}\gamma. \tag{3}$$

As shown by Misra *et al* and also in this paper, it is advantageous mathematically to work on the Ernst-like MPST equation to derive new solutions. But one must keep in view the transformations (3) that must be made to arrive at the physical results. We make use of these transformations in one instance only, namely, to show the identity of Bonnor's three-parameter solution (1979) with ours in § 4.1.

2. The MPST equation

A brief outline of the derivation of the MPST equation is presented here.

Let us consider a static axially symmetric metric given by

$$\mathrm{d}s^2 = f\,\mathrm{d}t^2 - \frac{\mathrm{e}^{2k}}{f}[\mathrm{d}\rho^2 + \mathrm{d}z^2] - \frac{\rho^2}{f}\,\mathrm{d}\varphi^2 \tag{4}$$

where f and k are functions of ρ and z.

Let us define the electromagnetic field as follows

$$F^{31} = \frac{f}{\rho}\mathrm{e}^{-2k}A_{,2} \qquad F^{23} = \frac{f}{\rho}\mathrm{e}^{-2k}A_{,1} \tag{5a}$$

$$F_{01} = B_{,1} \qquad F_{02} = B_{,2} \tag{5b}$$

where $A(\rho, z)$ and $B(\rho, z)$ are respectively the magnetic and electric potentials. Misra *et al* consider situations such that A is proportional to B in the following way

$$A = C\cos\epsilon \qquad B = C\sin\epsilon \tag{6}$$

where C is a new electromagnetic potential and ϵ is a constant. Now Misra *et al* introduce a complex function ξ defined by

$$\frac{\xi - 1}{\xi + 1} = f^{1/2} + \mathrm{i}C. \tag{7}$$

In terms of ξ the Einstein–Maxwell equations are expressed as

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\nabla\xi\cdot\nabla\xi \tag{8}$$

$$\frac{\partial k}{\partial\rho} = \frac{4\rho}{(\xi\xi^* - 1)^2}\left(\frac{\partial\xi}{\partial\rho}\frac{\partial\xi^*}{\partial\rho} - \frac{\partial\xi}{\partial z}\frac{\partial\xi^*}{\partial z}\right) \tag{9}$$

and

$$\frac{\partial k}{\partial\rho} = \frac{8\rho}{(\xi\xi^* - 1)^2}\,\mathrm{Re}\left(\frac{\partial\xi}{\partial\rho}\frac{\partial\xi^*}{\partial z}\right) \tag{10}$$

ξ^* being the complex conjugate of ξ. We shall refer to (8) as the MPST equation. It is Ernst-like (1968) in form.

3. A new two-parameter solution

Writing ε for $(\xi-1)/(\xi+1)$ the MPST equation (8) may be written in the following form

$$(\operatorname{Re}\varepsilon)\nabla^2\varepsilon=\nabla\varepsilon\cdot\nabla\varepsilon. \tag{11}$$

Now Tanabe (1979) has shown that from a known solution $\varepsilon=\kappa+\mathrm{i}h$ one can generate a new solution $\varepsilon'=\kappa'+\mathrm{i}h'$ in the following way. (11) may be rewritten in the form

$$(\operatorname{Re}\varepsilon)\rho^{-1}\nabla\cdot(\rho\nabla\varepsilon)=\nabla\varepsilon\cdot\nabla\varepsilon. \tag{12}$$

Since $\varepsilon=\kappa+\mathrm{i}h$ (12) is equivalent to the simultaneous equations

$$\nabla\cdot\left(\frac{\rho}{\kappa}\nabla\kappa\right)+\frac{\rho}{\kappa^2}\nabla h\cdot\nabla h=0 \tag{13}$$

$$\nabla\cdot\left(\frac{\rho}{\kappa^2}\nabla h\right)=0. \tag{14}$$

One can easily see that the new functions κ' and h' satisfy (13) and (14) if they are related to the known functions κ and h in the following manner

$$\kappa'=\rho/\kappa \tag{15}$$

$$\tilde{\nabla}h'=(\rho/\kappa^2)\nabla h \tag{16}$$

where

$$\tilde{\nabla}=\left(\frac{\partial}{\partial z},-\frac{\partial}{\partial\rho}\right). \tag{17}$$

In other words, if $\varepsilon=\kappa+\mathrm{i}h$ is a solution, then

$$\varepsilon'=\kappa'+\mathrm{i}jh' \tag{18}$$

is another solution, where j is a symbol having the properties (Tanabe 1978)

$$j^2=-1 \qquad j^*=j. \tag{19}$$

It is easy to check that the iteration of this operation twice yields the original equation

$$(\varepsilon')'=\varepsilon. \tag{20}$$

This means that we can obtain only one new solution from the old by the transformation $\varepsilon\to\varepsilon'$. By this operation f' and C' are given from (15) and (16) by

$$f'^{1/2}=\rho/f^{1/2} \tag{21}$$

$$\tilde{\nabla}C'=(\rho/f)\nabla C. \tag{22}$$

The MPST solution in spherical polar coordinates (Misra *et al* 1973) is

$$\mathrm{d}s^2=\left(\frac{r^2+e^2\cos^2\theta-2mr}{r^2+e^2\cos^2\theta}\right)^2\mathrm{d}t^2-\frac{(r^2-2mr+e^2\cos^2\theta)^2(r^2+e^2\cos^2\theta)^2}{(r^2-2mr+e^2\cos^2\theta+m^2\sin^2\theta)^3}$$
$$\times\left(\frac{\mathrm{d}r^2}{r^2-2mr+e^2}+\mathrm{d}\theta^2\right)-\frac{(r^2+e^2\cos^2\theta)^2(r^2-2mr+e^2)}{(r^2-2mr+e^2\cos^2\theta)^2}\sin^2\theta\,\mathrm{d}\varphi^2 \tag{23}$$

$$C=\frac{2me\cos\theta}{r^2+e^2\cos^2\theta} \tag{24}$$

where m is the mass of the object and $2me$ its electric or magnetic moment. The new solution has

$$f'^{1/2} = \frac{(r^2-2mr+e^2)^{1/2}(r^2+e^2\cos^2\theta)}{r^2-2mr+e^2\cos^2\theta}\sin\theta \tag{25}$$

$$C' = \frac{2mer\sin^2\theta}{r^2-2mr+e^2\cos^2\theta}. \tag{26}$$

The complete metric may be constructed easily. The asymptotic form of C' shows that it represents a monopolar charge with axial symmetry. Further (25) shows that the solution is not asymptotically flat. It appears that if the original solution is asymptotically flat, the new solution obtained by Tanabe's method (1979) misses this property.

4. Three-parameter solutions

4.1. The charged MPST solution

It is obvious that if ξ_0 is a solution of (8) then

$$\xi = e^{i\alpha}\xi_0 \tag{27}$$

is also a solution, where α is a real parameter.

Expressing (8) in spheroidal coordinates (x, y) which are related to cylindrical coordinates (ρ, z) by

$$\rho = \beta(x^2-1)^{1/2}(1-y^2)^{1/2} \tag{28}$$

$$z = \beta xy \tag{29}$$

β being a scale factor, Misra *et al* find the solution of (8) to be

$$\xi_0 = px + iqy \tag{30}$$

where $p^2+q^2=1$. Equations (9) and (10) are independent of α and so k is unchanged by the transformation (27).

From (27) together with (30) and (7) we obtain

$$f^{1/2} = 1-2\frac{px\cos\alpha - qy\sin\alpha + 1}{(px+1)^2+q^2y^2+2p(\cos\alpha-1)x-2qy\sin\alpha} \tag{31}$$

and

$$C = 2\frac{px\sin\alpha + qy\cos\alpha}{p^2x^2+q^2y^2+1+2(px\cos\alpha - qy\sin\alpha)}. \tag{32}$$

By a coordinate transformation $x=(r-m)/\beta$, $y=\cos\theta$ the metric can be expressed as

$$\begin{aligned} ds^2 = {} & \left(1-2\frac{mr+\gamma(\gamma-e\cos\theta)}{r^2+(e\cos\theta-\gamma^2)^2}\right)^2 dt^2 - \frac{(r^2-2mr+e^2\cos^2\theta-\gamma^2)^2}{[r^2-2mr+m^2\sin^2\theta+(e^2-\gamma^2)\cos^2\theta]^3} \\ & \times[r^2+(e\cos\theta-\gamma)^2]^2\left(\frac{dr^2}{r^2-2mr+e^2-\gamma^2}+d\theta^2\right) \\ & -\frac{[r^2+(e\cos\theta-\gamma)^2]^2(r^2-2mr+e^2-\gamma^2)}{(r^2-2mr+e^2\cos^2\theta-\gamma^2)^2}\sin^2\theta\, d\varphi^2 \end{aligned} \tag{33}$$

where m, e and γ are related to p, q, α and β by

$$p=\beta(m^2+\gamma^2)^{-1/2} \qquad q=e(m^2+\gamma^2)^{-1/2}$$

$$\cos\alpha=m(m^2+\gamma^2)^{-1/2} \qquad \sin\alpha=\gamma(m^2+\gamma^2)^{-1/2}$$

and

$$\beta^2=m^2+\gamma^2-e^2.$$

The electromagnetic potential will take the form

$$C=2\frac{(r-m)\gamma+em\cos\theta}{r^2+(e\cos\theta-\gamma)^2}. \tag{34}$$

(33) will reduce to the MPST solution if one puts $\gamma=0$.

Under transformation (3), (33) and (34) become

$$\begin{aligned}ds^2=&\left(1-2\frac{mr-\gamma(\gamma-e\cos\theta)}{r^2-(e\cos\theta-\gamma)^2}\right)^2 dt^2-\frac{(r^2-2mr-e^2\cos^2\theta+\gamma^2)^2}{[r^2-2mr+m^2\sin^2\theta-(e^2-\gamma^2)\cos^2\theta]^3}\\&\times[r^2-(e\cos\theta-\gamma)^2]^2\left(\frac{dr^2}{r^2-2mr-e^2+\gamma^2}+d\theta^2\right)\\&-\frac{[r^2-(e\cos\theta-\gamma)^2]^2(r^2-2mr-e^2+\gamma^2)}{(r^2-2mr-e^2\cos^2\theta+\gamma^2)^2}\sin^2\theta\, d\varphi^2\end{aligned} \tag{35}$$

$$C=2\frac{(r-m)\gamma+em\cos\theta}{r^2-(e\cos\theta-\gamma)^2}. \tag{36}$$

Recently Bonnor (1979) has given a three-parameter solution as follows

$$ds^2=-W^2[P^2Q^{-3}(Z^{-1}dr^2+d\theta^2)+ZP^{-2}\sin^2\theta\, d\psi^2]+P^2W^{-2}dt^2 \tag{37}$$

$$\varphi=W^{-1}[-e(r-\tfrac{1}{2}m)+mb\cos\theta] \tag{38}$$

where

$$\begin{aligned}P&=(r-\tfrac{1}{2}m)^2-a^2+b^2\sin^2\theta\\Q&=(r-\tfrac{1}{2}m)^2-a^2\cos^2\theta\\W&=r^2-(b\cos\theta+\tfrac{1}{2}e)^2\\Z&=(r-\tfrac{1}{2}m)^2-a^2\end{aligned} \tag{39}$$

and the arbitrary constants are related by

$$a^2=b^2+\tfrac{1}{4}(m^2-e^2). \tag{40}$$

Now, replacing $\frac{1}{2}e$, $\frac{1}{2}m$ and b by $-\gamma$, m and e respectively in equations (37)–(40), one will find that this solution is identical with our solution represented by (35) and (36). γ represents the charge of the dipole of moment $2me$.

4.2. Charged Weyl and Wang solutions

The Weyl family of solutions of (8) in spheroidal coordinates is given by

$$\xi_0=\frac{(x+1)^\delta+(x-1)^\delta}{(x+1)^\delta-(x-1)^\delta} \tag{41}$$

where the parameter δ takes only the positive integral values. Applying (27) to (41) and using (7) one can have

$$f^{1/2}=\frac{2(x^2-1)^\delta}{(x+1)^{2\delta}(1+\cos\alpha)+(x-1)^{2\delta}(1-\cos\alpha)} \tag{42}$$

$$C=\frac{[(x+1)^{2\delta}-(x-1)^{2\delta}]\sin\alpha}{(x+1)^{2\delta}(1+\cos\alpha)+(x-1)^{2\delta}(1-\cos\alpha)}. \tag{43}$$

Since (9) and (10) are unaffected by the transformation (27), we obtain

$$\mathrm{e}^{2k}=(x^2-1)^{\delta^2}/(x^2-y^2)^{\delta^2}. \tag{44}$$

The solution given by equations (42)–(44) is asymptotically flat. The asymptotic form of (42) is

$$C\approx\frac{2\delta\sin\alpha}{x}. \tag{45}$$

Wang (1974) obtained a solution of (8) in the form

$$\xi=\alpha/\beta \tag{46}$$

where α and β are complex polynomials of the spheroidal coordinates x and y. f and C are given by

$$f^{1/2}=A/B \tag{47}$$

and

$$C=D/B \tag{48}$$

where with $\alpha=u+\mathrm{i}v$ and $\beta=m+\mathrm{i}n$,

$$\begin{aligned} A&=u^2+v^2-m^2-n^2\\ B&=(u+m)^2+(v+n)^2\\ D&=2(vm-un) \end{aligned} \tag{49}$$

and k is given by

$$\mathrm{e}^{2k}=\frac{A^4}{p^{8\delta}(x^2-y^2)^{\delta^2}}. \tag{50}$$

Applying (27) to (46) and using (7) we obtain

$$f^{1/2}=1-2\frac{m^2+n^2+(um+vn)\cos\alpha+(un-vm)\sin\alpha}{u^2+v^2+m^2+n^2+2(um+vn)\cos\alpha+2(un-vm)\sin\alpha} \tag{51}$$

$$C=2\frac{(um+vn)\sin\alpha+(vm-un)\cos\alpha}{u^2+v^2+m^2+n^2+2(um+vn)\cos\alpha+2(un-vm)\sin\alpha}. \tag{52}$$

k will remain unchanged and hence is given by (50).

The asymptotic form of (52) when $q=0$ is

$$C\approx\frac{2\sin\alpha}{x}. \tag{53}$$

Thus equations (50)–(52) represent the gravitational field of a body possessing a charge together with an electric or magnetic moment. It may easily be seen that this metric is asymptotically flat.

4.3. Charged all-multipole solution

We have recently obtained a solution of (8) for an object having all kinds of multipoles in the form (Krori and Chaudhury 1981)

$$\xi_0 = (p+\mathrm{i}q)(g+1)/(g-1) \tag{54}$$

where

$$g = \exp[2/(y-x)] \tag{55}$$

with $p^2+q^2=1$.

From (54), (27) and (7) we obtain

$$f^{1/2} = \frac{2g}{(p\cos\alpha - q\sin\alpha + 1)(g^2-1)+2} \tag{56}$$

$$C = \frac{(g^2-1)(p\sin\alpha + q\cos\alpha)}{(p\cos\alpha - q\sin\alpha+1)(g^2-1)+2}. \tag{57}$$

e^{2k} will remain unchanged under the transformation (27). $\alpha=0$ will lead to our solution. The asymptotic form of C when $q=0$ is

$$C \approx -\frac{\sin\alpha}{x}. \tag{58}$$

Thus the constant α may be interpreted as a parameter associated with the charge of the object. The asymptotic flatness of the metric can be seen easily.

Acknowledgment

The authors express their profound gratitude to the Government of Assam, Dispur, for all facilities provided at Cotton College Gauhati-781001, India and at the Institute of Advanced Study in Science and Technology, Gauhati-781001, India to work out this paper.

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